Runge-Kutta Methods

\[ y'(t) = f(t, y), \quad t \in [a, b], \quad y(a) = \alpha \quad \text{(IVP)} \]

Fortunately, there are several ways to achieve the higher order l.t.e. of the Taylor methods without the need to evaluate \( f', f'', \ldots \). The most popular single-step IVP solvers currently in use are the Runge-Kutta (R-K) methods.

Recall that \( m = (y_{k+1} - y_k)/h \) is the ideal value for the quantity \( \phi(t_k, y_k) \) in the single-step method \( w_{k+1} = w_k + h\phi(t_k, w_k) \). Now view \( m \) as the average value of \( y' \) over the interval \( [t_k, t_{k+1}] \). The R-K perspective is to approximate \( m \) by averaging approximate samples of \( y' \) from this interval. Euler’s method is the R-K method which approximates the average value of the slope on \( [t_k, t_{k+1}] \) by an approximation to its value at the left endpoint:

\[ m \approx y'(t_k) \approx f(t_k, w_k) \equiv \phi_{\text{Euler}}. \]

If we think the left endpoint is not as representative as the midpoint, we can use

\[ m \approx y'(t_k + h/2) \approx f(t_k + \frac{h}{2}, w_k + \frac{h}{2}f(t_k, w_k)) \equiv \phi_{\text{Midpoint}}. \]

Maybe an average of the left and right endpoints seems better

\[ m \approx \frac{1}{2}(y'(t_k) + y'(t_k + h)) \approx \frac{1}{2}(f(t_k, w_k) + f(t_k + h, w_k + hf(t_k, w_k))) \equiv \phi_{\text{ModEuler}}. \]

These last two (Midpoint and Modified Euler) methods have l.t.e. \( O(h^2) \) and require 2 \( f \)-evals per iteration. R-K methods all have the form:

\[ \phi_{\text{RK}} = \sum_{i=1}^{m} r_i f(t_k + \tau_i, K_i), \]

where \( \tau_i \in [0, h] \) and \( K_i \) is another sum of \( f \)-values evaluated (hopefully) near the solution graph in \( [t_k, t_{k+1}] \).

For now we avoid the question of how to construct the weighted average of nested function evaluations that give a method with a certain l.t.e., but in order to do that one needs to determine where to sample the slope field (which \( \tau_i \) and \( K_i \) above), and what weights to give these values (which \( r_i \)).

There has been a tremendous amount of work done in this area, and we will end this too brief introduction by stating a theorem which tells us the best possible l.t.e. that can be achieved by a R-K method using \( n \) function evaluations per step.

**Theorem:**

<table>
<thead>
<tr>
<th># f-evals per step</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5, 6, 7</th>
<th>8, 9</th>
<th>10, 11</th>
</tr>
</thead>
<tbody>
<tr>
<td>best possible l.t.e.</td>
<td>( O(h^2) )</td>
<td>( O(h^3) )</td>
<td>( O(h^4) )</td>
<td>( O(h^{n-1}) )</td>
<td>( O(h^{n-2}) )</td>
<td>( O(h^{n-3}) )</td>
</tr>
</tbody>
</table>