Lagrange Interpolation on the Roots of Unity

Let $n$ be a positive integer. Define $x_k = e^{2\pi ik/n}$, $k = 0, 1, \ldots, n-1$, where $i = \sqrt{-1}$. Notice that $x_k^n = 1$, and as such these numbers are called the $n^{th}$ roots of unity. They are evenly spaced around the unit circle $|z| = 1$ in $\mathbb{C}$: (De Moivre)

$$x_k = e^{2\pi ik/n} = (\cos (2\pi/n) + i \sin (2\pi/n))^k = \cos (2\pi k/n) + i \sin (2\pi k/n).$$

Now suppose we wish to interpolate the data $(x_k, y_k)$, $k = 0, 1, \ldots, n-1$ with a polynomial $p(x) = \sum_{j=1}^{n-1} a_j x^j$ of degree $n - 1$ (the Lagrange interpolator).

The Vandermonde view says that $p$ can be determined by solving the system

$$Va = y,$$

where

$$a = [a_0, a_1, \ldots, a_{n-1}]^t, \quad y = [y_0, y_1, \ldots, y_{n-1}]^t$$

and

$$e_k^t V = [1, x_k, x_k^2, \ldots, x_k^{n-1}].$$

Now let’s investigate the (Hermitian) matrix $V^t V = V^* V = V^t V = [s_{kj}]$. If $k \neq j$,

$$s_{kj} = \sum_{r=0}^{n-1} x_k^r x_j^r = \sum_{r=0}^{n-1} e^{2\pi(j-k)r/n} = \sum_{r=0}^{n-1} (e^{2\pi(j-k)/n})^r = \frac{(e^{2\pi(j-k)/n})^{n} - 1}{e^{2\pi(j-k)/n} - 1} = 0,$$

and for $k = j$,

$$s_{jj} = \sum_{r=0}^{n-1} 1^r = n.$$

so $V^* V = nI$. But then $V^* Va = V^* y$, so the coefficients of $p$ are

$$a = V^{-1} y = \left(\frac{1}{n}\right)V^* y.$$

What’s the big deal? Well, by De Moivre, this is a discrete Fourier transform:

$$p(x) = \sum_{r=0}^{n-1} a_r (\cos (2\pi r/n) + i \sin (2\pi r/n)).$$

and the $a’s$ are the DFT coefficients.

This DFT, as described here, is simply matrix multiplication by $V^*$, and requires $O(n^2)$ flops, but taking advantage of the special structure of $V$ leads to

$$a = \text{FFT}(y),$$

requiring only $O(n \log(n))$ flops. Polynomials and trigonometric functions famously meet here, and at the Chebyshev polynomials, in some of the most fundamental and elegant of classical mathematics.