Monte Carlo Integration

When you don’t know what's going on, there is always statistics...

The average value of a function $f$ over the interval $[a, b]$ is defined by

$$(b - a) f_{\text{ave}} = \int_a^b f(x) \, dx$$

(width * height = area). If $f$ is a fcn of 2 variables the average value of $f$ over $\Omega$ is

$${\text{area}(\Omega)} f_{\text{ave}} = \int_{\Omega} f(x,y) \, dA$$

(area * height = volume). For higher dimensions, we have

$${\text{volume}(\Omega)} f_{\text{ave}} = \int_{\Omega} f(X) \, dV,$$

where we interpret “volume” in the general sense (and this may not be easy to compute). If $\Omega$ is a “rectangular” region, then volume($\Omega$) is easy to compute, and we write:

$$\left[ \prod_{j=1}^{k} (b_j - a_j) \right] f_{\text{ave}} = \int_{a_k}^{b_k} \cdots \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x_1, x_2, \ldots, x_k) \, dx_1 \, dx_2 \cdots dx_k.$$

So what? Well, we can turn these definitions around to get expressions for the definite integral, and lacking the true value $f_{\text{ave}}$, we can sample from the domain to get an approximate value for $f_{\text{ave}}$ and therefore an approximate value for the integral. If we sample the domain by choosing $x$ randomly, then the technique is called Monte Carlo integration.

If we sample from the domain uniformly ($x_j \in \Omega$ is just as likely to be selected as any other $x_k \in \Omega$), and then compute the sample average

$$\hat{f} = \frac{1}{N} \sum_{i=1}^{N} f(x_i),$$

then we have the integral approximation

$$\int_{\Omega} f \, dV \approx \text{volume}(\Omega) \, \hat{f}.$$

How good is our approximation? Since the variance of the sum of $N$ identically distributed independent r.v.’s is a factor of $1/N$ smaller than the variance of just one, and since the central limit theorem says that as $N \rightarrow \infty$, we can interpret this variance as that of a Gaussian distribution, we can therefore consider the error in a Monte Carlo simulation with $N$ samples to behave (statistically) like $O(1/\sqrt{N})$.

Notice that $N$ is the number of function evaluations, and comparing this error estimate to other quadrature rules we have developed, we see that Monte Carlo integration is extremely inefficient for low dimensional quadratures. This technique is an important tool in scientific computation, but is only used for very high dimensional problems.