Order of Convergence

The 'Big-O' notation is used to give an idea of the rate of convergence, but is often insufficient to convey how fast fast convergence can be. For quickly converging sequences, the order of convergence does a much better job. $p_n \to p$ of order $\alpha$ if there is a $\lambda > 0$ such that

$$\lim_{n \to \infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lambda.$$ 

The number $\lambda$ is called the asymptotic error constant. In the context of numerical methods, we think of $e_n \equiv p_n - p$ as an error, (and certainly much less than 1), and for large $n$ we should expect

$$|e_{n+1}| \approx \lambda |e_n|^\alpha.$$ 

It should be clear that since $p_n \to p$, then $\alpha \geq 1$. By the same token, if $\alpha = 1$, then $\lambda < 1$, in fact, $\alpha = 1$, $\lambda < 1$ corresponds to an exponential rate of convergence given by $\beta_n = \lambda^n = 1/(1/\lambda)^n$. This is a convergence rate that we thought was fast, we call it a linear order of convergence, but now consider $\alpha > 1$ .

If $\alpha = 2$ and $\lambda = 1$, then for large $n$, $|e_{n+1}| \approx |e_n|^2$. For example, if $e_3 = 0.01$, then $e_4 \approx 0.0001$, $e_5 \approx 10^{-8}$, and $e_6 \approx 10^{-16}$. This is called a quadratic ($\alpha = 2$) order of convergence, and in this case the number of correct digits approximately doubles at each iteration. What about the number of correct digits in a cubically ($\alpha = 3$) convergent sequence?

If $\alpha > 1$, the order of convergence is called superlinear (this is technically correct, but if $\alpha > 2$ we say superquadratic, so in practice superlinear means $1 < \alpha < 2$). Superlinear convergence is exhibited by some very important methods, and we study it here a bit. We say that a sequence which behaves as

$$\lim_{n \to \infty} \frac{|p_{n+1} - p|}{|p_n - p|} = 0$$

is superlinear (even though you may not be able to find any such $\alpha > 1$).

Now superlinear convergence guarantees

$$\lim_{n \to \infty} \frac{|p_{n+1} - p_n|}{|p_n - p|} = \lim_{n \to \infty} \frac{|p_{n+1} - p + p - p_n|}{p_n - p} = \lim_{n \to \infty} \frac{|p_{n+1} - p| + p - p_n}{p_n - p} = 1.$$ 

Which says that for large enough $n$, we get a computable error estimate

$$|e_n| = |p_n - p| \approx |p_{n+1} - p_n|$$