The Power Method

Assume that $A \in \mathbb{C}^{n \times n}$ has $n$ linearly independent eigenvectors $v_1, v_2, \ldots, v_n$. Then any $x \in \mathbb{C}^n$ can be represented uniquely as

$$x = \sum_{i=1}^{n} c_i v_i. \quad (1)$$

Here we are interested in what (if any) direction $A^k x$ heads toward as $k \to \infty$. Specifically, we have a sequence $\{x_k\}$ of vectors defined by

$$x_0 = x, \quad x_k = A x_{k-1} = A^k x_0, \quad k = 1, 2, 3, \ldots \quad (2)$$

and we would like to know in what direction it is ultimately pointing.

Recall that if $v_i$ is an eigenvector of $A$, then there is a scalar $\lambda_i$, called an eigenvalue, for which $Av_i = \lambda_i v_i$. Then $A^k v_i = \lambda_i^k v_i$ (you do the induction). Using (1) (and linearity) we find that

$$A^k x = \sum_{i=1}^{n} c_i \lambda_i^k v_i. \quad (3)$$

Now suppose that $|\lambda_1| > |\lambda_i|, \ i = 2, 3, \ldots, n$. Then

$$\frac{A^k x}{\lambda_1^k} = c_1 v_1 + \sum_{i=2}^{n} c_i \left( \frac{\lambda_i}{\lambda_1} \right)^k v_i \quad (4)$$

Here it is clear (yes?) that unless $c_1 = 0$, $A^k x \to \text{span}\{v_1\}$. Thus we call $v_1$ the dominant eigenvector of $A$. This result is as simple as it is powerful: if $v_1$ is the dominant eigenvector of $A$, then for almost all $x \in \mathbb{C}^n$,

$$x \to v_1$$

under repeated application of $A$.

(If this is too analytic for your taste, then change to the basis $\{v_1, v_2, \ldots, v_n\}$. Under this basis $A$ has coordinates $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, and $A^k y \to \text{span}\{e_1\}$ as long as $y(1) \neq 0$.)

The power method consists of scaling iteration (2) to avoid underflow or overflow, and figuring out when to stop. We solve both problems by approximating $\lambda_1$ at each step. The code below (if it terminates) gives a small backward error (i.e. gives an eigenpair for a matrix “close” to $A$).

$$i = \text{argmax}(|x|)$$
$$x = x / x(i)$$

For $k = 1, 2, \ldots$ until done

- $y = Ax$
- $i = \text{argmax}(|y|)$
- $\lambda = y(i)$
- $r = y - \lambda x$, if $\|r\|$ is small enough, then stop
- $x = y / \lambda$