General instructions:

(1) Do any four out of the five problems. If you choose to do five problems, we will not pick the best four out of your five solutions.

(2) Begin the solution to each problem by explaining, briefly, your approach to solving the problem. This will count for 5 points. The rest of the problem solving will count for 95 points.

(3) Write your I. D. code (not your name) and the problem number on each of your answer sheets.

(4) Use only one side of the paper for your answer. Furthermore, do not write your answers near the edge of the paper otherwise part(s) of your answer may not reproduce in photocopying.

(5) Hand in each problem as a unit. Number your pages. Turn in your formula sheet with your solutions. Make sure that your name is NOT on the formula sheet.
Problem 1

An electric charge distribution is described by the following charge density that exists inside a spherical region of radius \( a \) and vanishes outside

\[
\rho = \rho_0 \left[ \cos^2 \theta - \frac{1}{3} \right].
\]

(a) Find the potential everywhere (both inside and outside the charge distribution).

(b) Now suppose that we surround the charge distribution in part (a) by a grounded conducting sphere of radius \( b > a \) centered at the origin. Find the new potential everywhere inside the sphere \((0 < r < b)\). Check that the new potential is continuous across \( r = a \).

You may find the following formulas in the spherical coordinate system useful.

\[
\Phi(r) = \frac{1}{\epsilon_0} \sum_{\ell, m} \frac{Y_{\ell m}(\theta, \phi)}{2\ell + 1} \int_{r'^{\ell+1}}^{r^{\ell+1}} \rho(r') d^3r', \quad \Phi(r) = \sum_{\ell, m} \left[ A_{\ell m} r^\ell + B_{\ell m} r^{\ell+1} \right] Y_{\ell m}(\theta, \phi)
\]

\[
Y_{00} = \frac{1}{\sqrt{4\pi}}, \quad Y_{10} = \sqrt{\frac{3}{4\pi}} \cos \theta, \quad Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi},
\]

\[
Y_{20} = \sqrt{\frac{5}{4\pi}} \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right), \quad Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\phi}, \quad Y_{22} = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin \theta e^{2i\phi},
\]

\[
Y_{\ell 0} = \sqrt{\frac{2\ell + 1}{4\pi}} P^\ell_{-1} \left( \cos \theta \right), \quad Y_{\ell,-m} = (-)^m Y_{\ell m}^*.
\]

Problem 2

(a) Consider the displacement current \( j_d = \epsilon_0 \frac{\partial E}{\partial t} \) and use Maxwell’s equations to show that the total current \( J = j + j_d \), where \( j \) is the current due to free charges, is divergence-less, i.e., \( \nabla \cdot J = 0 \). (20 points)

(b) A conducting sphere of radius \( a \) is being charged through a straight wire, carrying a current \( I \), so that the charge on the sphere obeys \( \dot{q} = I \). Assuming a symmetric distribution of charges over the surface of the sphere, find the electric field outside the sphere. Determine also the displacement current, and verify the conservation law \( \nabla \cdot J = 0 \). (30 points)

(c) Using Ampere-Maxwell law in the form \( \nabla \times B = \mu_0 J \) or its integral version, find the magnetic field everywhere outside the sphere. [40 points]

(d) By appropriately limiting your result from (c), verify that close to the wire, the answer leads to the familiar form for an infinitely long straight wire carrying a current \( I \). (10 points)
Problem 3

Two large metal plates are held a distance $d$ apart, one at ground potential zero and the other at potential $V_0$. A solid metal hemisphere (radius $a \ll d$) is placed near the center of the grounded plate so that its potential is likewise zero. If the region between the plates is filled with a weakly conducting material of uniform conductivity $\sigma$, what current flows to the hemisphere?

Problem 4

A monochromatic plane wave moving in a nondispersive medium of refractive index $n_1 \approx 1$, is incident at an angle $\theta_i$ with the normal to the interface of semi-infinite ionosphere of dielectric constant $\varepsilon_2/\varepsilon_0 \approx [1 - (\omega_p/\omega)^2]$, where $\omega_p = N e^2/m_e \varepsilon_0$ is the plasma frequency, $N$ is the number of electrons per unit volume. The wave is polarized perpendicular to the plane of incidence with its complex electric field given by

$$E_i(r, t) = E_0 e^{i(k_1 r - \omega t)}.$$

(a) Find the frequency dependence of the refractive index of the ionosphere assuming $\mu_1 = \mu_2 = \mu_0$. What happens to the wave propagating in the ionosphere when $\omega < \omega_p$ and $\omega > \omega_p$.

(b) Write down the fields for incident, reflected, and transmitted waves and the boundary conditions they satisfy. Using these boundary conditions, derive Fresnel’s equations for $\omega > \omega_p$.

(c) Find the coefficients of reflection $R$ and transmission $T$ and show that $R + T = 1$. Show that beyond a critical angle of incidence, the wave is totally reflected back into the medium of incidence. Find this critical angle and comment upon its frequency dependence.

(d) A radio amateur operating at a wavelength of 15.0 m at night can receive radio signals from 800 km or more but not from any locations closer than this. Assuming that the ionosphere is 300 km above the ground, estimate the plasma frequency.

Problem 5

A spring with force constant $k$ is hanging from the ceiling. A particle of mass $M$ and charge $Q$ is attached to this spring and is at height $h$ above the floor. The mass is pulled down by a distance $d$ below the equilibrium and released.

(a) Calculate radiation fields and the power radiated per unit solid angle in the long wavelength limit.

(b) Calculate the intensity of radiation hitting the floor at a distance $R$ from the point directly below the charge $Q$ in the radiation zone $[d \ll \lambda \ll h]$.

(c) Find the $R$ for which the radiation is most intense?

(d) Find the average energy per unit time hitting the ground assuming the floor is infinite.